# *WORCESTER COUNTY MATHEMATICS LEAGUE*

## Varsity Meet 4 - March 2, 2016
## Round 1: Number Theory

*All answers must be in simplest exact form in the answer section*

**NEITHER CALCULATOR NOR RULER ALLOWED**

1. What is the last digit of the number you get by multiplying the first 2016 odd prime numbers together?

2. Compute: $6^{2026}$ mod 37.

3. Emma invented a way to encrypt natural numbers. For each natural number, she first converts the number into base-5 form, and then uses each of the letters V, W, X, Y and Z to represent one specific digit in the base-5 form. Emma notices that the letter representations for three consecutive natural numbers are VYZ, VYX and VVW (increasing in that order).
What is the natural number (in base-10) that will be encrypted as VWXYZ (in base-5)?

**ANSWERS**

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

## *WORCESTER COUNTY MATHEMATICS LEAGUE*

**Varsity Meet 4 - March 2, 2016**
**Round 2: Algebra I**

*All answers must be in simplest exact form in the answer section*
**NO CALCULATOR ALLOWED**

1. Solve for $x$ and $y$:

$$2x + y = -7$$
$$x + 2y = -2$$

2. If $x = 1 + 2^{p}$ and $y = 1 + 2^{-p}$, express $y$ in terms of $x$.
Express answer in rational form.

3. Solve for $x$:

$$(x^2 - x - 1)^{3x^2+5x-2} = 1$$

**ANSWERS**

(1 pt.) 1. ______________________________

(2 pts.) 2. ______________________________

(3 pts.) 3. ______________________________

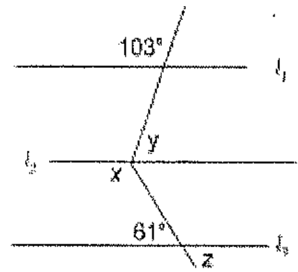
# *WORCESTER COUNTY MATHEMATICS LEAGUE*

## Varsity Meet 4 - March 2, 2016
## Round 3: Geometry

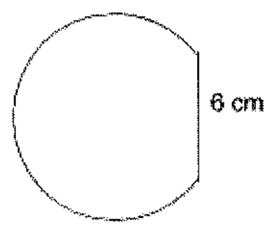
*All answers must be in simplest exact form in the answer section*

**NO CALCULATOR ALLOWED**

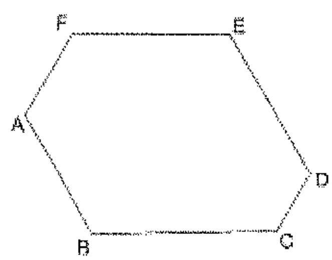
1. Find the value of $\frac{x-y}{z}$, where $l_1$, $l_2$, $l_3$ are parallel.




2. A portion of a circle is removed as shown in the figure on the right. If the length of AB is 6 cm and the area of the circle is $12\pi$, find the area of the figure. (Not drawn to scale.)




3. As shown, ABCDEF is an irregular hexagon with $\angle A = \angle B = \angle C = \angle D = \angle E = \angle F$. Given AB + BC = 11 and FA – CD = 3. (Not drawn to scale.) Compute the value of BC + DE.




## ANSWERS

(1 pt.) 1. ______________________________

(2 pts.) 2. ______________________________ $cm^2$

(3 pts.) 3. ______________________________

# *WORCESTER COUNTY MATHEMATICS LEAGUE*

## Varsity Meet 4 - March 2, 2016
## Round 4: Logarithms, Exponents, and Radicals

*All answers must be in simplest exact form in the answer section*

**NO CALCULATOR ALLOWED**

1. Solve for $x$:

$$\sqrt[4]{5x-2} = 2$$

2. Express $y$ in terms of $a$ and $x$ in the simplest form:

$$3\log_x a - \log_x y = 3$$

3. Solve for $x$:

$$\left(\frac{x}{9}\right)^{\log 9} = \left(\frac{x}{11}\right)^{\log 11}$$

**ANSWERS**

(1 pt.) 1. ________________________

(2 pts.) 2. ________________________

(3 pts.) 3. ________________________

# *WORCESTER COUNTY MATHEMATICS LEAGUE*

## Varsity Meet 4 - March 2, 2016
## Round 5: Trignometry

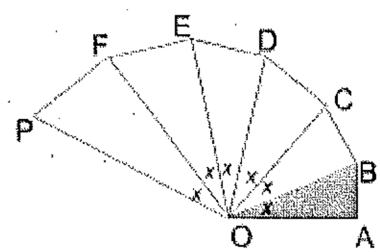
*All answers must be in simplest exact form in the answer section*

**NO CALCULATOR ALLOWED**

1. The length of a pendulum is 18 cm. Find, in terms of $\pi$, the distance through which the tip of the pendulum travels when the pendulum turns through an arc of 120°.

2. Solve for $x$: $\cos^2 x - \sin x = \frac{1}{4}$, $0 \le x < 2\pi$

3. As the graph on the right shows, starting from line segment AO, draw a right triangle BOA with BO being the hypotenuse. Then, use BO as a leg and draw a new right triangle BOC, with CO being the hypotenuse and $\angle AOB = \angle BOC$. If we repeat the process for six times in total (and thus generate six right triangles), the length of the last hypotenuse is $PO = \frac{216}{125} \cdot AO$. Compute $\sin(x)$.




## ANSWERS

(1 pt.) 1. ______________________ cm

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________

# *WORCESTER COUNTY MATHEMATICS LEAGUE*

## Varsity Meet 4 - March 2, 2016

## Team Round

*All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (3 points each)*

**APPROVED CALCULATORS ALLOWED**

1. Let $Q$ be the largest whole number that is less than 169 and has the same number of factors as 169. Write $Q$ in base-8 form.

2. Solve for $x$:

$$\left(27^{\frac{2}{3}}\right)^{3x} = 243 \bullet 3^{2x}$$

3. Find the positive difference in area between a circle with radius $r$ and a regular hexagon inscribed in the circle with radius $r$.

4. There are three cards, and the numbers on the cards are all distinct and smaller than 10. The cards are distributed to three people, and then recollected and redistributed again in random. If the sums of the numbers on each person's cards are 10, 11 and 13, what are the numbers on the three cards?

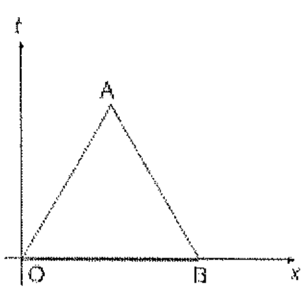
5. Consider $\triangle ABC$ with $\angle CAB = 105^\circ$, and $\angle ABC$ is $\frac{3}{2}$ times the measure of $\angle BCA$. If AC = 8. find the length of AB.

6. The geometry we have learned is called Euclidean Geometry, which defines the distance between point $(x_1, y_1)$ and $(x_2, y_2)$ as

$$d = \sqrt{\left|(x_1 - x_2)^2 + (y_1 - y_2)^2\right|}$$

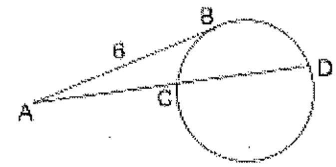
In Einstein's relativity, the geometry is different because the distance between two points is defined differently. If we label point A with $(t_1, x_1)$ and point B with $(t_2, x_2)$, the distance AB is defined by

$$d = \sqrt{\left|-(t_1 - t_2)^2 + (x_1 - x_2)^2\right|}$$




Under such relativity geometry, calculate the perimeter of the equilateral triangle on the right. The length of each edge is 6 units under Euclidean Geometry.

7. As the graph shows, line segment AB is tangent to the circle at point B, and line AD intersects the circle at point C and D. Given that the length of AB is 6. If the length of AD and CD are both integers, what is the smallest possible value of $|AC - CD|$?




8. Define sets A, and B on all positive integers. If set A represents the set of all odd numbers, and if set B represents the set of all prime numbers, use set operation (union, intersection and complement) to express the set {2}

9. Find the area of a convex quadrilateral with vertices A(1, 5), B(–1, 2), C(1, –1) and D(3, 2).

# *WORCESTER COUNTY MATHEMATICS LEAGUE*

## Varsity Meet 4 - March 2, 2016 Answer Key

**Round 1:**
1. 5 (Algonquin)
2. 36
3. 1358

**Round 2:**
1. $x = -4$, $y = 1$ or $(-4, 1)$ (Doherty)
2. $y = \frac{x}{x-1}$ (Bromfield)
3. $-2$, $-1$, 0, $\frac{1}{3}$, 1 and 2 (in any order)

**Round 3:**
1. $\frac{42}{61}$ or $\frac{42°}{61°}$ (Burncoat)
2. $8\pi + 3\sqrt{3}$ (Bancroft)
3. 14

**Round 4:**
1. 3.6 or $\frac{18}{5}$ or $3\frac{3}{5}$ (Bromfield)
2. $\frac{x^3}{a^3}$
3. 99 (Hudson)

**Round 5:**
1. $12\pi$ (Nipmuc)
2. $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ (Assabet)
3. $\frac{\sqrt{6}}{6}$

**TEAM Round:**
1. 171 (Quaboag)
2. $\frac{5}{4}$ or 1.25 (Doherty)
3. $\pi r^2 - \frac{3\sqrt{3}}{2}r^2$ or $(\pi - \frac{3\sqrt{3}}{2})r^2$ (Trans Babb)
4. 4, 6 and 7 (in any order)
5. $4\sqrt{2}$ (Assabet)
6. $6 + 6\sqrt{2}$
7. 1
8. $A^c \cap B$ or $B \cap A^c$
9. 12 (St. Peter-Marian)

# ***WORCESTER COUNTY MATHEMATICS LEAGUE***

**Varsity Meet 4 - March 2, 2016**
**Team Round Answer Sheet**

1. ______________________ (base 8)
2. ______________________
3. ______________________
4. ______________________
5. ______________________
6. ______________________
7. ______________________
8. ______________________
9. ______________________

# *WORCESTER COUNTY MATHEMATICS LEAGUE*

## Varsity Meet 4 - March 2, 2016 - Solutions
## Round 1: Number Theory

*All answers must be in simplest exact form in the answer section*

**NO CALCULATOR AND RULER ALLOWED**

1. What is the last digit of the number you get by multiplying the first 2016 odd prime numbers together?

**Solution**: Notice that 5 is one of the prime numbers. The product of the 2016 prime numbers must be divisible by 5, and therefore the last digit must be 5 or 0. Since all the prime numbers are odd, their product must also be odd, and the last digit must be 5.

2. Compute: $6^{2026}$ mod 37.

**Solution**: First, we know that $6^2 = 36$, so $6^{2026} = 36^{1013}$. Then, observe that 36 mod 37 = –1. In modulo, the residual of the product divided by a number is the same as the product of the residual divided by the same number. Therefore,

$$\begin{aligned} & 6^{2026} \bmod 37 \\ &= 36^{1013} \bmod 37 \\ &= (36 \bmod 37)^{1013} \\ &= (-1)^{1013} \\ &= -1 \end{aligned}$$

In modulo, the answer has to be a positive number. If a number mod 37 is equal to –1, that means the number need to increase by 1 to become a multiple of 37. We find that such number would be 36, which is an acceptable value in modulo.

3. Emma invented a way to encrypt natural numbers. For each natural number, she first converts the number into base-5 form, and then uses each of the letters V, W, X, Y and Z to represent one specific digit in the base-5 form. Emma notices that the letter representations for three consecutive natural numbers are VYZ, VYX and VVW (increasing in that order).
What is the natural number (in base-10) that will be encrypted as VWXYZ (in base-5)?

**Solution** : Since VYZ, VYX and VVW are consecutive in base-5 form, first we observe that Z is immediately followed by X. That is,

$$Z + 1 = X$$

When VYX is increased by 1, the number changes into VVW, which means (VYX + 1) gives a different tenths digit. The only possibility is that $X = 4_5$, and $X + 1 = 4_5 + 1_5 = 10_5$. Therefore,

$$X = 4$$
$$Z = X - 1 = 3$$

Also, since X + 1 gives a 0 as the last digit, and since VYX + 1 = VVW, we have

$$W = 0$$

Then, Y and V can only take numbers 1 and 2. Since VYX + 1 = VVW and the sum of the unit digits is $10_5$, the tenths digit must increase by 1. Therefore, V = Y + 1, and we can observe that V = 2 and Y = 1.

Therefore, VWXYZ = $20413_5$. Convert that back to base-10,

$$\begin{aligned} &20413_5 \\ &= 2 \cdot 5^4 + 0 \cdot 5^3 + 4 \cdot 5^2 + 1 \cdot 5^1 + 3 \cdot 5^0 \\ &= 2 \cdot 625 + 4 \cdot 25 + 5 + 3 \\ &= 1250 + 100 + 5 + 3 \\ &= 1358 \end{aligned}$$

# *WORCESTER COUNTY MATHEMATICS LEAGUE*

## Varsity Meet 4 - March 2, 2016 - Solutions
## Round 2: Algebra I

*All answers must be in simplest exact form in the answer section*

**NO CALCULATOR ALLOWED**

1. Solve for $x$ and $y$:

$$2x + y = -7$$
$$x + 2y = -2$$

**Solution**: We can multiply the first equation by 2,

$$4x + 2y = -14$$

Then, subtract the second equation from the first equation to eliminate $y$,

$$(4x + 2y) - (x + 2y) = (-14) - (-2)$$
$$4x + 2y - x - 2y = -14 + 2$$
$$3x = -12$$
$$x = -4$$

Plug the result into the first equation,

$$2(-4) + y = -7$$
$$y = -7 - 2(-4) = -7 + 8 = 1$$

2. If $x = 1 + 2^p$ and $y = 1 + 2^{-p}$, express $y$ in terms of $x$. Express answer in rational form.

**Solution**: From the first formula,

$$x = 1 + 2^p$$
$$2^p = x - 1$$

From the second formula,

$$y = 1 + 2^{-p}$$
$$2^{-p} = y - 1$$
$$2^p = \frac{1}{y-1}$$

Equate the two formula,

$$\frac{1}{y-1} = 2^p = x-1$$
$$(x-1)(y-1) = 1$$
$$y-1 = \frac{1}{x-1}$$
$$y = \frac{1}{x-1} + 1 = \frac{x}{x-1}$$

3. Solve for $x$:

$$(x^2 - x - 1)^{3x^2+5x-2} = 1$$

**Solution:** In order for the formula to hold, we can either have $x^2 - x - 1 = 1$ or $3x^2 + 5x - 2 = 0$, or $x^2 - x - 1 = -1$ where $3x^2 + 5x - 2$ is an even number.

In the first case, we have

$$x^2 - x - 1 = 1$$
$$x^2 - x - 2 = 0$$
$$(x+1)(x-2) = 0$$
$$x = -1 \text{ or } x = 2$$

In the second case, we have

$$3x^2 + 5x - 2 = 0$$
$$(3x-1)(x+2) = 0$$
$$x = \frac{1}{3} \text{ or } x = -2$$

In the third case, we have

$$x^2 - x - 1 = -1$$
$$x^2 - x = 0$$
$$x = 1 \text{ or } x = 0$$

And to check that $3x^2 + 5x - 2$ is an even number,

$$3(1)^2 + 5(1) - 2 = 6$$
$$3(0)^2 + 5(0) - 2 = -2$$

As we can see, whenever $x^2 - x - 1 = -1$, $3x^2 + 5x - 2$ is an even number, so both $x = 1$ and $x = 0$ can solve the problem.

So we obtain six solutions for the equation:
$x = -1$, $x = 2$, $x = \frac{1}{3}$, $x = 0$, $x = 1$ and $x = -2$.

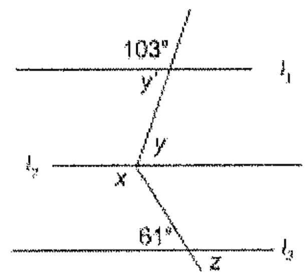
# WORCESTER COUNTY MATHEMATICS LEAGUE

## Varsity Meet 4 - March 2, 2016 - Solutions
## Round 3: Geometry

*All answers must be in simplest exact form in the answer section*

**NO CALCULATOR ALLOWED**

1. Find the value of $\frac{x-y}{z}$, where $l_1$, $l_2$, $l_3$ are parallel.

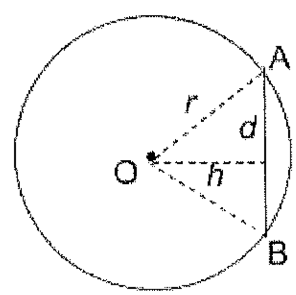



**Solution**: Since $l_1$ and $l_2$ are parallel, $y$ and $y'$ has the same angle. Since $y'$ and the 103° angles are along the same line, they are supplements, and thus $y = y' = 180° - 103° = 77°$.

For $x$, observe that $l_2$ and $l_3$ are parallel. Therefore, $x$ and the 61° angles are supplements, and thus $x = 180° - 61° = 119°$. Also $z = 61°$.

Therefore, the final answer is $\frac{x-y}{z} = \frac{119°-77°}{61°} = \frac{42}{61}$

2. A portion of a circle is removed as shown in the figure on the right. If the length of AB is 6 cm and the area of the circle is $12\pi$, find the area of the figure. (Not drawn to scale.)




**Solution** : As the graph shows, draw the arc AB. Denote the center of the circle by O, and connect AO and BO. Then, draw a line from point O perpendicular to AB, and denote the line segment by $h$. Denote the upper part of line segment AB by $d$.

If we want to find the area of the figure, we have to first find the area enclosed by chord AB and arc AB. Observe that the enclosed area is the difference between fan OAB and triangle OAB. To find the area of that weird shape, we can first find the area of the fan OAB, and subtract the area of triangle OAB from the area of the fan.

First, since AO = BO because they are both the radii of the circle, and since $h$ is perpendicular to AB, $h$ bisects AB, and thus

$$d = \frac{1}{2} \bullet 6 \text{ cm} = 3 \text{ cm}$$

Then, since the area of the circle is $12\pi$, by the formula of the area of a circle, we have

$$\pi r^2 = 12\pi \text{ cm}^2$$
$$r^2 = 12 \text{ cm}^2$$
$$r = 2\sqrt{3} \text{ cm}$$

Then, by pythagorean theorem, we can compute that

$$h^2 = r^2 - d^2 = 12 - 9 = 3 \text{ cm}^2$$
$$h = \sqrt{3} \text{ cm}$$

Notice that this is a special 1-$\sqrt{3}$-2 triangle. $2h = 2\sqrt{3} = r$, and $\sqrt{3}h = 3 = d$. The length ratio of the three edges are thus $h$-$d$-$r$ = $h$-$\sqrt{3}h$-$2h$ = 1-$\sqrt{3}$-2. Therefore, the angle between AO and $h$ is 60°. Also, we can calculate the area of triangle OAB, being

$$\tfrac{1}{2}h \bullet AB = \tfrac{1}{2} \bullet \sqrt{3} \bullet 6 = 3\sqrt{3} \text{ cm}^2$$

Since the angle AOB is twice of the angle between AO and $h$, $\angle$AOB = 120°, which is a third of 360°. Therefore, the area of the gan AOB is also a third of the area of the circle, which is $4\pi$ cm$^2$. Therefore, the area enclosed by the chord AB and arc AB is

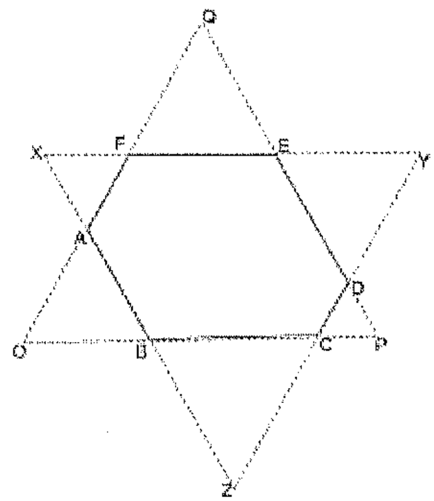
$$4\pi - 3\sqrt{3} \text{ cm}^2$$

And thus the area of the figure is the difference between the area of the circle and the area above, which is

$$12\pi - (4\pi - 3\sqrt{3}) = 8\pi + 3\sqrt{3} \text{ cm}^2$$

3. As shown, ABCDEF is an irregular hexagon with $\angle A = \angle B = \angle C = \angle D = \angle E = \angle F$. Given AB + BC = 11 and FA – CD = 3. Compute the value of BC + DE.

**Solution**: As the graph shows, we can extend each of the edges of the irregular hexagon and form two triangles. Since $\angle A = \angle B = \angle C = \angle D = \angle E = \angle F$, and since we know that each interior angle of a regular hexagon is 120°, we can claim that the six angles here are all 120°. If one angle is larger than 120°, since the inner angles of a polygon always sum up to the same number, another angle must be smaller than 120° to compensate, and the condition will not hold.




Now, observe that triangle OPQ and XYZ are all equilateral triangles. To prove that, since $\angle A = \angle B = 120°$, we can compute that $\angle OAB = \angle OBA = 60°$, and thus $\angle O = 60°$. Similarly, we can derive that

$$\angle O = \angle P = \angle Q = \angle X = \angle Y = \angle Z = 60°$$

Therefore, triangle OPQ and XYZ must both be equilateral triangles. Also, one can observe that the six small triangles around the hexagon are all equilateral as well.

Since triangle OAB is equilateral, OB = AB. Similarly, triangle PCD is equilateral, and CP = CD. Therefore, since AB + BC = 11, the length of the edge of triangle OPQ is

$$OP = OB + BC + CP = AB + BC + CD = 11 + CD$$

Similarly, the length of the edge of triangle XYZ is

$$XZ = XA + AB + BZ = FA + AB + BC = 11 + FA$$

Since FA – CD = 3, FA = 3 + CD, and the edge of triangle XYZ is

$$XZ = 11 + 3 + CD = 14 + CD$$

Now, we can map BC to CZ and DE to DY, and get

$$BC + DE = ZC + CD + DY = ZY - CD = XY - CD = XZ - CD = 14$$

# *WORCESTER COUNTY MATHEMATICS LEAGUE*

## Varsity Meet 4 - March 2, 2016 - Solutions
## Round 4: Logarithms, Exponents, and Radicals

*All answers must be in simplest exact form in the answer section*

**NO CALCULATOR ALLOWED**

1. Solve for $x$:

$$\sqrt[4]{5x-2} = 2$$

**Solution**: Raise both sides to the fourth power, and we get

$$5x - 2 = 2^4 = 16$$
$$5x = 16 + 2 = 18$$
$$x = \tfrac{18}{5}$$

2. Express $y$ in terms of $a$ and $x$ in the simplest form:

$$3 \log_x a - \log_x y = 3$$

**Solution:** First, we want to change the bases of all the logarithms to the same one. Let's pick $\log_a$. Then, since

$$3 \cdot log_x a = \frac{3log_a a}{log_a x} = \frac{3}{log_a x}$$
$$log_x y = \frac{log_a y}{log_a x}$$

The original equation becomes

$$\frac{3}{log_a x} - \frac{log_a y}{log_a x} = 3$$
$$3 - log_a y = 3 \cdot log_a x$$
$$log_a y = 3 - 3 \cdot log_a x$$
$$y = a^{(3-3 \cdot log_a x)}$$
$$= a^3/(a^{3log_a x})$$
$$= a^3/(a^{log_a x^3}) = \frac{a^3}{x^3}$$

3. Solve for $x$:

$$\left(\frac{x}{9}\right)^{\log 9} = \left(\frac{x}{11}\right)^{\log 11}$$

**Solution**: First, take log on both sides, and get

$$\log\left(\frac{x}{9}\right)^{\log 9} = \log\left(\frac{x}{11}\right)^{\log 11}$$

We can move the power down to the front of each term because it is logarithm, which gives

$$\log 9 \bullet \log\left(\frac{x}{9}\right) = \log 11 \bullet \log\left(\frac{x}{11}\right)$$
$$\log 9 \bullet (\log x - \log 9) = \log 11 \bullet (\log x - \log 11)$$
$$\log 9 \bullet \log x - (\log 9)^2 = \log 11 \bullet \log x - (\log 11)^2$$

Rearrange all the terms, we get

$$(\log 11 - \log 9) \bullet \log x = (\log 11)^2 - (\log 9)^2$$

By the formula of square difference,

$$(\log 11 - \log 9) \bullet \log x = (\log 11)^2 - (\log 9)^2$$
$$= (\log 11 - \log 9) \bullet (\log 11 + \log 9)$$

$$\log x = \log 11 + \log 9 = \log 99$$
$$x = 99$$

# ***WORCESTER COUNTY MATHEMATICS LEAGUE***

## Varsity Meet 4 - March 2, 2016 - Solutions
## Round 5: Trigonometry

*All answers must be in simplest exact form in the answer section*

**NO CALCULATOR ALLOWED**

1. The length of a pendulum is 18 cm. Find, in terms of $\pi$, the distance through which the tip of the pendulum travels when the pendulum turns through an arc of 120°.

**Solution 1**: The area swept by the pendulum is fan-shaped with a 120° angle, and the length swept by the tip of the pendulum is thus the arc of the fan-shaped area. The circumference of a circle with radius 18 cm is $18 \bullet 2\pi = 36\pi$ cm. Since a 120° fan shape is $\frac{120^\circ}{360^\circ} = \frac{1}{3}$ of a circle, the length of the arc of the fan shape is thus also $\frac{120^\circ}{360^\circ} = \frac{1}{3}$ of the circumference of the circle. The result is $\frac{1}{3} \times 36\pi = 12\pi$ cm.

**Solution 2**: One can compute that $120^\circ = 120^\circ \times \frac{\pi}{180^\circ} = \frac{2}{3}\pi$, so the length of the arc corresponding to the 120° angle is $S = r\theta = 18 \times \frac{2}{3}\pi = 12\pi$ cm.

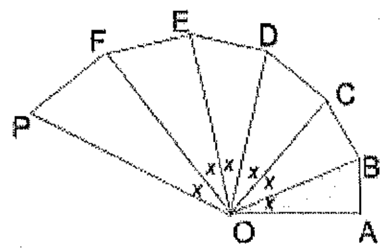
2. Solve for $x$: $\cos^2 x - \sin x = \frac{1}{4}$, $0 \le x < 2\pi$

**Solution**: Since $\cos^2 x + \sin^2 x = 1$, we have

$$\cos^2 x - \sin x = \tfrac{1}{4}$$
$$1 - \sin^2 x - \sin x = \tfrac{1}{4}$$
$$-\sin^2 x - \sin x = \tfrac{3}{4}$$
$$4\sin^2 x + 4\sin x - 3 = 0$$
$$(2\sin x + 3)(2\sin x - 1) = 0$$
$$\sin x = \tfrac{-3}{2} \text{ or } \sin x = \tfrac{1}{2}$$

Since the value of $\sin x$ cannot be larger than 1 or smaller than $-1$, $\sin x = \frac{1}{2}$ is the only solution. The corresponding values of $x$ are thus $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

3. As the graph on the right shows, starting from line segment AO, draw a right triangle BOA with BO being the hypotenuse. Then, use BO as a leg and draw a new right triangle BOC, with CO being the hypotenuse and $\angle AOB = \angle BOC$. If we repeat the process for six times in total (and thus generate six right triangles), the length of the last hypotenuse is PO = $\frac{216}{125} \bullet AO$. Compute sin($x$).




**Solution**: We can apply trig to $\triangle AOB$ and compute that $\cos x = \frac{AO}{BO}$. Similarly, we can apply trig to $\triangle BOC$ and derive $\cos x = \frac{BO}{CO}$. Since

$$\angle AOB = \angle BOC = \angle COD = \angle DOE = \angle EOF = \angle FOP = x$$

We can apply the same process and deduce that

$$\cos x = \frac{AO}{BO} = \frac{BO}{CO} = \frac{CO}{DO} = \frac{DO}{EO} = \frac{EO}{FO} = \frac{FO}{PO}$$

Observe that

$$\frac{AO}{PO} = \frac{AO}{BO} \bullet \frac{BO}{CO} \bullet \frac{CO}{DO} \bullet \frac{DO}{EO} \bullet \frac{EO}{FO} \bullet \frac{FO}{PO} = \cos^6 x$$

Since PO = $\frac{216}{125} \bullet AO$, we can derive that

$$\cos^6 x = \frac{AO}{PO} = \frac{125}{216}$$

$$\cos x = \sqrt{\frac{5}{6}}$$

$$\sin x = \sqrt{1 - cos^2 x} = \sqrt{1 - \frac{5}{6}} = \sqrt{\frac{1}{6}} = \sqrt{\frac{6}{36}} = \frac{\sqrt{6}}{6}$$

# *WORCESTER COUNTY MATHEMATICS LEAGUE*

## Varsity Meet 4 - March 2, 2016 - Solutions

## Team Round

*All answers must be in simplest exact form, unless stated otherwise. (3 points each)*

**APPROVED CALCULATORS ALLOWED**

1. Let $Q$ be the largest whole number that is less than 169 and has the same number of factors as 169. Write $Q$ in base-8 form.

**Solution**: Since $169 = 13 \bullet 13$, the number 169 has three factors. Note that, in order for a compound number to have three factors, it must be the square of a prime number. Otherwise, the compound number should contain at least four factors.

The largest square of a prime number below 169 is $11^2 = 121$, so $Q = 121$ is the largest number that satisfy the condition.

To write 121 in base-8 form, $121 = 64 + 57 = 64 + 7 \bullet 8 + 1 = 8^2 + 7 \bullet 8 + 1 = 171_8$.

2. Solve for $x$:

$$\left(27^{\frac{2}{3}}\right)^{3x} = 243 \bullet 3^{2x}$$

**Solution** : Notice that $27 = 3^3$ and $243 = 3^5$, so we have

$$[(3^3)^{\frac{2}{3}}]^{3x} = 3^5 \times 3^{2x}$$

$$(3^2)^{3x} = 3^{5+2x}$$

$$(3)^{6x} = 3^{5+2x}$$

$$6x = 5 + 2x$$

$$4x = 5$$

$$x = \frac{5}{4} = 1.25$$

3. Find the positive difference in area between a circle with radius $r$ and a regular hexagon inscribed in the circle with radius $r$.

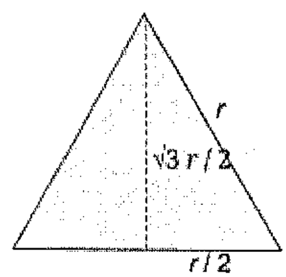
**Solution**: We need to calculate the area of the hexagon and the circle, and then subtract one from the other.

The area of a circle with radius *r* is, by formula,

$$\pi r^2$$

To compute the area of a regular hexagon, first note that the hexagon is inscribed in a circle with radius *r*. We know that, for inscription, the center of the hexagon is the center of the circle, and the six vertices are on the circumference of the circle. Therefore, the distance from the center of the hexagon to each of its vertices is also *r*.

Another observation is that, after we connect the center of the hexagon with its six vertices, we divide the hexagon into six equilateral triangles, and the center of the hexagon is a vertex of each of the six triangles. Since the lines connecting the center of the hexagon and its vertices are also the edges of the triangles, the length of the edge of each of the six equilateral triangles is *r*. We can thus compute each triangle's area, and sum them up together to get the total area, which is the area of the hexagon.




The graph on the right shows an equilateral triangle with edge *r*. Draw a height, and the height must divide the base equally into two parts. Therefore, the length of each part is $\frac{r}{2}$. Since the height is perpendicular to the base and the equilateral triangle has an angle of 60 degrees, the right half of the equilateral triangle is a special 1-$\sqrt{3}$-2 triangle. Therefore, the length of the height is

$$\sqrt{3} \cdot \frac{r}{2} = \frac{\sqrt{3}}{2} r$$

Therefore, the area of each such equilateral triangle is

$$\frac{1}{2} \cdot r \cdot \frac{\sqrt{3}}{2} r = \frac{\sqrt{3}}{4} r^2$$

Since there are six such triangles in a hexagon, the area of the hexagon is

$$\frac{\sqrt{3}}{4} r^2 \cdot 6 = \frac{3\sqrt{3}}{2} r^2$$

And the area difference is

$$\pi r^2 - \frac{3\sqrt{3}}{2} r^2$$

4. There are three cards, and the numbers on the cards are all distinct and smaller than 10. The cards are distributed to three people, and then recollected and redistributed again in random. If the sums of the numbers on each person's cards are 10, 11 and 13, what are the numbers on the three cards?

**Solution 1**: Since each time all of the three cards have to be drawn, the total sum of the three people's numbers must be a multiple of the sum of the three cards. Consider

$$10 + 11 + 13 = 34 = 17 \cdot 2$$

The sum of the three cards has to be either 17 or 2. The sum of the numbers on three cards cannot be 2 because it is too small. Therefore, the sum of the numbers of the three cards is 17, and each card is picked up exactly twice in total. We can discuss the problem in different cases where a person picks up the same card twice or not.

If a person picked up a card twice, the sum of two of the same number must be even, so only the first person could pick up the same card twice, and the other two people must pick up different cards. Since there are only two kinds of cards available, each of the other two people has to pick up each card twice, which implies that the sum of their numbers must be the same, but it is not the case in the question. Therefore, each person has to pick up different cards.

As we are sure that each person picks up different cards, we know each of the three people's numbers are the sum of two numbers, and we also know the sum of all three numbers. If we subtract the sum of two numbers from the sum of three numbers, we can get the value of the number that is not included in the first sum. Therefore,

$$17 - 10 = 7$$
$$17 - 11 = 6$$
$$17 - 13 = 4$$

are the three numbers on the cards.

**Solution 2**: We can also apply a guess-and-check method to pin down the values of the cards. First, as Solution 1, we derive that the sum of the values on the three cards is 17. Now we can try the values one by one.

If one of the cards is 9, no one could pick up that card twice because 18 is larger than the three existing sums, so two people have to each picks up the card once. Since we already have the sum for all three people, the other two cards have to take two of the three values $10 - 9 = 1$, $11 - 9 = 2$ or $13 - 9 = 4$. There could be 3 combinations in total. However, since the sum of all cards is 17, the other two cards must sum to 8, and that means the three combinations must be too small.

If one of the cards is 8, for similar reason that does not work. if one of the cards is 7, the sum of the other two cards is 10. After some trials, one can discover that the 4 & 6 combination solves the problem. So the three cards are 4, 6 and 7.

5. Consider △ABC with ∠CAB = 105°, and ∠ABC is $\frac{3}{2}$ times the measure of ∠BCA. If AC = 8. find the length of AB.

**Solution:** As the graph shows, draw line AD perpendicular to the base BC. We can assume the measure of ∠BCA is $x$, and the measure of ∠ABC is thus 1.5$x$. Since ∠CAB = 105°, and since the sum of all the inner angles of a triangle is 180°, we can solve for $x$:


$$x + 1.5x + 105^\circ = 180^\circ$$
$$2.5x = 180^\circ - 105^\circ = 75^\circ$$
$$x = 30^\circ$$

The measure of ∠BCA is 30°.

Since ∠ADC is a right angle, triangle ADC is a special right triangle with an angle equal to 30°. Therefore, the length of AD is

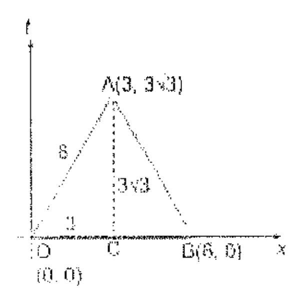
$$AD = AC \div 2 = 8 \div 2 = 4$$

Similarly, ∠ABC = 1.5$x$ = 45°, so triangle ABD is a special right triangle with a 45° angle. Therefore, the length of AB is

$$AB = \sqrt{2} \cdot AD = \sqrt{2} \cdot 4 = 4\sqrt{2}$$

6. The geometry we have learned is called Euclidean Geometry. In Einstein's relativity, the geometry is different because the distance between two points are defined differently. If we label point A with ($t_1$, $x_1$) and point B with ($t_2$, $x_2$), the distance AB is defined by

$$d = \sqrt{\left| -(t_1 - t_2)^2 + (x_1 - x_2)^2 \right|}$$



Under such relativity geometry, calculate the perimeter of the equilateral triangle on the right. The length of each edge is 6 units under Euclidean Geometry.

**Solution:** Notice that the perimeter of the triangle is the sum of the distances from O to A, O to B and A to B. We can first compute the coordinates of point O, A and B, and then apply the new distance formula to compute the three distances and add them up together.

Since point O is put at the origin, its coordinate is (0, 0). Point B is along the $x$-axis, so its $t$ coordinate is 0. Since the Euclidean length of OB is 6, the coordinate of B is (6, 0). To compute the coordinate of point A, first draw a line segment AC perpendicular to OB. Since triangle OAB is equilateral, AC bisects OB and OC = 3. Then, since the triangle is equilateral, $\angle AOC = 60^\circ$, and we have a special right triangle AOC with a $60^\circ$ angle. Therefore,

$$AC = \sqrt{3} \cdot OC = \sqrt{3} \cdot 3 = 3\sqrt{3}$$

And we know the coordinate of A is $(3,\ 3\sqrt{3})$.

Then, we can apply the formula to compute the distances:

$$OA = \sqrt{|-(t_A - t_O)^2 + (x_A - x_O)^2|} = \sqrt{|-(3\sqrt{3} - 0)^2 + (3-0)^2|}$$
$$= \sqrt{|-27+9|} = \sqrt{18} = 3\sqrt{2}$$

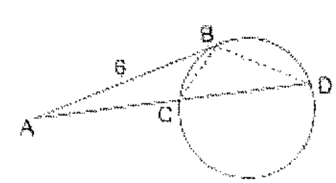
$$OB = \sqrt{|-(t_B - t_O)^2 + (x_B - x_O)^2|} = \sqrt{|-(0-0)^2 + (6-0)^2|}$$
$$= \sqrt{36} = 6$$

$$AB = \sqrt{|-(t_A - t_B)^2 + (x_A - x_B)^2|} = \sqrt{|-(3\sqrt{3} - 0)^2 + (3-6)^2|}$$
$$= \sqrt{|-27+9|} = \sqrt{18} = 3\sqrt{2}$$

Therefore, the parameter is the sum of the three distances, which is

$$OA + OB + AB = 3\sqrt{2} + 6 + 3\sqrt{2} = 6 + 6\sqrt{2}$$

7. As the graph shows, line segment AB is tangent to the circle at point B, and line AD intersects the circle at point C and D. Given that the length of AB is 6. If the length of AD and CD are both integers, what is the smallest possible value of $|AC - CD|$?



**Solution:** First, derive the numerical relationship between AB, AC and CD. This uses a classic secant tangent theorem, $AB^2 = AC \bullet AD$. First, to see the relationship, connect BC and BD. $\angle ABC = \angle BDC$ if AB is tangent to the circle, and if CD is a chord. Also, $\angle BAC = \angle DAB$ because they are the same angle. Therefore, $\triangle ABC$ and $\triangle ADB$ have two angles that have the same magnitude, and the two triangles are thus similar to each other. By such similarity, we derive

$$AB : AD = AC : AB$$
$$AB^2 = AC \bullet AD$$

Since AC and CD both have integer length, the length of AD must also be an integer. Since $AB^2 = 36$, the length of AC and AD must both be the factors of 36. We can try to break down 36 into the product of different numbers and test them one by one. Note that the length of AD cannot be smaller than the length of AC because AD contains AC. Then,

a) $36 = 1 \bullet 36$, so AC = 1, AD = 36. CD = AD – AC = 35, and $|AC-CD| = |1 - 35| = 34$
b) $36 = 2 \bullet 18$, so AC = 2, AD = 18. CD = AD – AC = 16, and $|AC-CD| = |2 - 16| = 14$
c) $36 = 3 \bullet 12$, so AC = 3, AD = 12. CD = AD – AC = 9, and $|AC-CD| = |3 - 9| = 6$
d) $36 = 4 \bullet 9$, so AC = 4, AD = 9. CD = AD – AC = 5, and $|AC-CD| = |4 - 5| = 1$
e) $36 = 6 \bullet 6$, so AC = 6, AD = 6. CD = AD – AC = 0, and $|AC-CD| = |0 - 6| = 6$

Therefore, when AC = 4 and CD = 5, we attain the smallest value 1.

8. Define sets A, B and C on all positive integers. If set A represents the set of all odd numbers, set B represents the set of all prime numbers, and set C represents the set of all compound numbers, use set operation (union, intersection and complement) to express the set {1, 2}.

**Solution:** We start the problem from the special property of 2. Since 2 is the only even prime number, {2} can be expressed as the intersection of all prime numbers and all even numbers. The set of all even numbers is the complement of set A, so $\{2\} = A^c \cap B$.

9. Find the area of a convex quadrilateral with vertices A(1, 5), B(–1, 2), C(1, –1) and D(3, 2).

**Solution 1**: We need to first notice that such quadrilateral is a rhombus. We can compute the length of the four edges and find that

$$\begin{aligned}
AB &= \sqrt{(1-(-1))^2+(5-2)^2} = \sqrt{2^2+3^2} = \sqrt{13}\\
BC &= \sqrt{(-1-1)^2+(2-(-1))^2} = \sqrt{2^2+3^2} = \sqrt{13}\\
CD &= \sqrt{(1-3)^2+(-1-2)^2} = \sqrt{2^2+3^2} = \sqrt{13}\\
DA &= \sqrt{(3-1)^2+(2-5)^2} = \sqrt{2^2+3^2} = \sqrt{13}
\end{aligned}$$

Therefore, the length of the four edges is equal, and the quadrilateral is a rhombus. To calculate the area of a rhombus, note that the area is half of the product of the two diagonal lines. The diagonal lines are AC and BD because these two pair of points are opposite to each other. We can compute the length of the two lines and get

$$\begin{aligned}
AC &= \sqrt{(1-1)^2+(5-(-1))^2} = \sqrt{0^2+6^2} = 6\\
BD &= \sqrt{(-1-3)^2+(2-2)^2} = \sqrt{4^2+0^2} = 4
\end{aligned}$$

And the area is thus

$$\frac{1}{2} \cdot AC \cdot BD = \frac{1}{2} \cdot 6 \cdot 4 = 12$$

**Solution 2**: Notice that the diagonals are perpendicular. A(1, 5) and C(1, –1) form a vertical line of length 6, while B(–1, 2) and D(3, 2) form a horizontal line of length 4. The Quadrilateral can be divided into four right triangles with legs of $x$ or $6-x$ and $y$ or $4-y$. The four possible combinations of such two sets of possible values of legs define the legs of the four right triangles:

Triangle 1: $\frac{1}{2}xy$
Triangle 2: $\frac{1}{2}x(4-y) = \frac{1}{2}(4x-xy)$
Triangle 3: $\frac{1}{2}(6-x)y = \frac{1}{2}(6y-xy)$
Triangle 4: $\frac{1}{2}(6-x)(4-y) = \frac{1}{2}(xy-4x-6y+24)$

Add up the areas of the four triangles, we can get the total area of the quadrilateral, which is

$$\begin{aligned}
&\frac{1}{2}[xy+(4x-xy)+(6y-xy)+(xy-4x-6y+24)]\\
&= \frac{1}{2}(24+4x-4x+6y-6y+2xy-2xy)\\
&= 12
\end{aligned}$$